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# Factors of Regular Graphs

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A spanning subgraph $F$ of a graph $G$ is called a $[k-1, k]$-factor if $k-1 \leqslant d_F(x) \leqslant k$ for all vertices $x$ of $G$, where $d_F(x)$ denotes the degree of $x$ in $F$. Tutte proved that if $r$ is an odd integer, then every $r$-regular graph has a $[k-1, k]$-factor for every integer $k$, $0<k<r$. We prove that if $r$ is odd and $0<k \leqslant 2r/3$, then every $r$-regular graph has a $[k-1, k]$-factor each of whose components is regular. © 1986 Academic Press, Inc.

## 1. INTRODUCTION

We consider a finite graph $G$ which may have multiple edges but has no loops. A graph without multiple edges is called a *simple graph*. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of $G$, respectively. We write $d_G(x)$ for the degree of a vertex $x$ in $G$. Let $a$, $b$, and $r$ be integers such that $0 \leqslant a \leqslant b$ and $r>0$. A spanning subgraph $F$ of $G$ is called an *$[a, b]$-factor* of $G$ if $a \leqslant d_F(x) \leqslant b$ for all $x \in V(G)$, and we usually call an $[r, r]$-factor an *$r$-factor*. An *$r$-regular graph* is a graph in which each vertex has degree $r$. Other notation and definitions not defined in this paper can be found in [3 or 4].

Tutte [12] ([4, p. 77]) proved that for any odd integer $r$ and any integer $k$ $(0<k<r)$, every $r$-regular graph has a $[k-1, k]$-factor. It was proved in [6, 11] that every regular graph has a $[1, 2]$-factor each of whose components is regular. Enomoto and Saito [5] gave the following conjecture: Every $r$-regular graph has a $[k-1, k]$-factor each of whose components is regular for any $k$, $0<k<r$. Note that this conjecture is true when $r$ is even by Petersen's 2-factorable theorem (see Lemma 1). So the essential part of this conjecture is the case that $r$ is odd. Main results of this paper are the following two theorems, and a theorem on $[a, b]$-factors is given in Section 3.

THEOREM 1. *Let $r$ and $k$ be positive integers. If $k \leqslant 2(2r+1)/3$, then every $(2r+1)$-regular graph has a $[k-1, k]$-factor each of whose components is regular.*



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THEOREM 2. *Let $k$ and $r$ be positive integers. If $2r+2-\sqrt{2r+2}<k\leqslant 2r$, then there exists a simple $(2r+1)$-regular graph that has no $[k-1, k]$-factor each of whose components is regular.*

Some results related to our theorems can be found in a survey article [1].

## 2. PROOFS OF THEOREMS 1 AND 2

Let $G$ be a graph, and $g$ and $f$ be integer-valued functions defined on $V(G)$ such that $g(x)\leqslant f(x)$ for all $x\in V(G)$. A spanning subgraph $F$ of $G$ is called a *$(g,f)$-factor* if $g(x)\leqslant d_F(x)\leqslant f(x)$ for all $x\in V(G)$. A $(g,f)$-factor satisfying $g(x)=f(x)$ for all $x\in V(G)$ is briefly called an *$f$-factor*. For a vertex subset $X$ of $G$, we write $G-X$ for the subgraph of $G$ obtained from $G$ by deleting the vertices in $X$ together with their incident edges. Similarly, for an edge subset $Y$ of $G$, $G-Y$ denotes the subgraph of $G$ obtained from $G$ by deleting all the edges in $Y$. For two disjoint subsets $S$ and $T$ of $V(G)$, we denote by $e_G(S, T)$ or $e(S, T)$ the number of edges of $G$ joining $S$ and $T$.

LEMMA 1 (Petersen [10]; [3] p. 166). *Every $2r$-regular graph has a $2k$-factor for every integer $k$, $0<k<r$.*

LEMMA 2 (Lovász [9]). *Let $G$ be a graph, and $g$ and $f$ be integer-valued functions defined on $V(G)$ satisfying $g(x)\leqslant f(x)$ for all $x\in V(G)$. Then $G$ has a $(g,f)$-factor if and only if*

$$\delta(S, T)=\sum_{t\in T}(d_G(t)-g(t))+\sum_{s\in S}f(s)-e_G(S, T)-h(S, T)\geqslant 0$$

*for all disjoint subsets $S$ and $T$ of $V(G)$, where $h(S, T)$ denotes the number of components $C$ of $G-(S\cup T)$ such that $g(x)=f(x)$ for all $x\in V(C)$ and*

$$\sum_{x\in V(C)} f(x)+e_G(V(C), T)\equiv 1 \qquad (\text{mod } 2).$$

Lemma 2 plays an important role through this paper, and its short proof is given in [13]. The next lemma is obtained by using Tutte's $f$-factor theorem [11].

LEMMA 3 [7]. *Let $G$ be an $n$-edge-connected graph ($n\geqslant 1$), $\theta$ be a real number such that $0\leqslant\theta\leqslant 1$, and $f$ be an integer-valued function defined on $V(G)$. If (1), (2), and (3) hold, then $G$ has an $f$-factor:*

(1) $\sum_{x \in V(G)} f(x) \equiv 0 \pmod 2$.

(2) $\varepsilon = \sum_{x \in V(G)} |f(x) - \theta d_G(x)| < 2$.

(3) $\{f(x) \mid x \in V(G)\}$ *consists of even numbers, and* $m(1-\theta) \geqslant 1$, *where* $m \in \{n, n+1\}$ *and* $m \equiv 1 \pmod 2$.

LEMMA 4 [2]. *Let $G$ be a 2-edge-connected $(2r+1)$-regular graph, and $h$ be a positive integer. If $2h \leqslant 2(2r+1)/3$, then $G$ has a $2h$-factor. If $(2r+1)/3 \leqslant 2h+1 \leqslant 2r+1$, then $G$ has a $(2h+1)$-factor. In particular, for every integer $k$, $0 < k \leqslant 2r+1$, $G$ has a $[k-1, k]$-factor each component of which is regular.*

Note that the above lemma can be proved by Lemma 3.

LEMMA 5. *Let $G$ be a 2-edge-connected $[2r, 2r+1]$-graph having exactly one vertex $w$ of degree $2r$. Then*

(1) *if $0 \leqslant 2k \leqslant 2(2r+1)/3$, then $G$ has a $2k$-factor; and*

(2) *if $(2r+1)/3 \leqslant 2k+1 \leqslant 2r+1$, then $G$ has a $[2k, 2k+1]$-factor $F$ such that $d_F(w) = 2k$ and $d_F(x) = 2k+1$ for all $x \in V(G) \setminus \{w\}$.*

*Proof.* We first prove (1). Define a function $f$ on $V(G)$ by $f(x) = 2k$ for all $x \in V(G)$, and put $\theta = 2k/(2r+1)$. Then $G$, $f$, and $\theta$ satisfy conditions (1), (2) ($\varepsilon = 2k/(2r+1) < 1$) and (3) of Lemma 3. Hence $G$ has an $f$-factor, which is a $2k$-factor.

We next prove (2). Let $F$ be a $2k$-factor given in (1). Then the complement $G - E(F)$ is a desired factor of (2).

Let $\{a, b, ..., n\}$ be a set of integers. A graph $G$ is called an $\{a, b, ..., n\}$-*graph* if $d_G(x) \in \{a, b, ..., n\}$ for all $x \in V(G)$. Analogously, we define an $\{a, b, ..., n\}$-*factor*. We denote an edge joining a vertex $x$ to a vertex $y$ by $xy$ or $yx$. The proof of Theorem 1 heavily depends on the next lemma.

LEMMA 6. *Let $G$ be a connected $(2r+1)$-regular graph with at least two bridges, and $k$ be a positive integer. If $(2r+4)/3 \leqslant k \leqslant 2(2r+1)/3$, then $G$ has a $[k-1, k]$-factor each component of which is regular.*

*Proof.* We first prove that if $(2r+1)/3 \leqslant 2k+1 \leqslant 2(2r+1)/3$, then $G$ has a $[2k, 2k+1]$-factor each of whose components is regular. Since $G$ has at least two bridges, it follows that for any bridge $vw$ of $G$, $G - vw$ has at most one 2-edge-connected component. Let $v_1 w_1, ..., v_n w_n$ be the bridges of $G$ such that $G - v_i w_i$ has exactly one 2-edge-connected component $C_i$ for each $i$, $1 \leqslant i \leqslant n$, where we assume $w_i \in V(C_i)$ and $v_i \in V(G) \setminus V(C_i)$. Put

$$H = G - \bigcup_{i=1}^{n} (V(C_i) \setminus \{w_i\}),$$

which is the subgraph of $G$ induced by the edge subset $E(G)\setminus\{E(C_1)\cup\cdots\cup E(C_n)\}$. It is immediate that $H$ is a connected $\{2r+1, 1\}$-graph with end vertices $w_1,\ldots, w_n$.

We shall prove that $H$ has a $\{2k+1, 1, 0\}$-factor $F$ such that $d_F(x)=2k+1$ if $d_H(x)=2r+1$, and $d_F(w)\in\{1, 0\}$ for all end vertices $w$. Define two functions $g$ and $f$ on $V(H)$ by

$$\begin{aligned} g(x) &= 2k+1 && \text{if} \quad d_H(x)=2r+1,\\ &=0 && \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} f(x) &= 2k+1 && \text{if} \quad d_H(x)=2r+1,\\ &=1 && \text{otherwise.} \end{aligned}$$

Since a $(g,f)$-factor of $H$ is a required $\{2k+1, 1, 0\}$-factor, it suffices to show that $\delta(S, T)\geqslant 0$ for all $S, T\subset V(H)$ such that $S\cap T=\varnothing$ by Lemma 2. Suppose $\delta(S, T)<0$ for some disjoint subsets $S$ and $T$ of $V(H)$. Since $H$ has at least two end vertices, $\delta(\phi, \phi)=-h(\phi, \phi)=0$. Hence $S\cup T\neq\varnothing$. Choose disjoint subsets $S$ and $T$ of $V(H)$ in such a way that $|S\cup T|$ is minimum subject to $\delta(S, T)<0$. Denote by $D_1,\ldots, D_m$ the components of $H-(S\cup T)$ which satisfy the conditions on $h(S, T)$ in Lemma 2.

We now prove that neither $S$ nor $T$ contains end vertices of $H$. Assume $S$ contains an end vertex $w$. Let $wz$ be the edge of $H$ incident with $w$. It follows that

$$\begin{aligned} \delta(S, T) &= \sum_{t\in T}(d_H(t)-g(t))+\sum_{s\in S-w} f(s)+1-e(S, T)-h(S, T)\\ &=\delta(S-w, T)+1-e(w, T)-h(S, T)+h(S-w, T). \end{aligned}$$

If $z\in T$, then $e(w, T)=1$ and $h(S, T)=h(S-w, T)$. Thus $\delta(S-w, T)=\delta(S, T)<0$, which contradicts the choice of $S$ and $T$. If $z\in S$, then $e(w, T)=0$ and $h(S, T)=h(S-w, T)$, and so $\delta(S-w, T)=\delta(S, T)-1<0$, a contradiction. If $z\in V(D_i)$ for some $D_i$, then $e(w, T)=0$ and $h(S-w, T)=h(S, T)-1$, and so $\delta(S-w, T)=\delta(S, T)<0$, a contradiction. If $z$ is contained in a component of $G-(S\cup T)$ that is not $D_i$ $(1\leqslant i\leqslant m)$, then $e(w, T)=0$ and $h(S-w, T)=h(S, T)$. Hence $\delta(S-w, T)<0$, a contradiction. Therefore $S$ does not contain any end vertex of $H$.

Suppose $T$ contains an end vertex $w$. Let $wz$ be the edge of $H$. Then it follows that

$$\delta(S, T)=\delta(S, T-w)+1-e(S, w)-h(S, T)+h(S, T-w).$$

If $z\in T$, then $\delta(S, T-w)=\delta(S, T)-1$, a contradiction. If $z\in S$, then $\delta(S, T-w)=\delta(S, T)$, a contradiction. If $z\in V(D_i)$ for some $D_i$, then

$\delta(S, T-w)=\delta(S, T)$, a contradiction. If $z$ is contained in a component of $H-(S\cup T)$ that is not $D_i$ $(1\leqslant i\leqslant m)$, then $\delta(S, T-w)=\delta(S, T)-1$, a contradiction. Therefore $T$ does not contain any end vertices of $H$. Consequently, $S\cup T$ contains no end vertices of $H$.

By the conditions on $D_i$ (see Lemma 2), we have that $d_H(x)=2r+1$ for all $x\in V(D_i)$. Moreover it follows from the conditions on $D_i$ that

$$\sum_{x\in V(D_i)} f(x)+e(V(D_i), T)\equiv |V(D_i)|+e(V(D_i, T)\equiv 1 \pmod 2. \quad (1)$$

It is trivial that $e(S\cup T, V(D_i))\geqslant 1$ as $S\cup T\neq\varnothing$. If $e(S\cup T, V(D_i))=1$, then the bridge joining $S\cup T$ to $D_i$ or a certain bridge of $D_i$ must be chosen in $\{v_1w_1,..., v_nw_n\}$, a contradiction. Therefore

$$e(S\cup T, V(D_i))\geqslant 2. \quad (2)$$

Let $h(S, T)=m$ and $\theta=(2r-2k)/(2r+1)$. Then $0<\theta<1$ and we obtain

$$\begin{aligned}\delta(S, T)&=(2r-2k)\,|T|+(2k+1)\,|S|-e(S, T)-h(S, T)\\ &=\frac{2r-2k}{2r+1}\sum_{t\in T} d_H(t)+\frac{2k+1}{2r+1}\sum_{s\in S} d_H(s)-e(S, T)-m\\ &\geqslant\theta\left\{e(T, S)+\sum_{i=1}^{m} e(T, V(D_i))\right\}\\ &\quad+(1-\theta)\left\{e(S, T)+\sum_{i=1}^{m} e(S, V(D_i))\right\}-e(S, T)-m\\ &=\sum_{i=1}^{m}\{\theta e(T, V(D_i))+(1-\theta)\,e(S, V(D_i))-1\}.\end{aligned}$$

Put

$$\Delta(D_i)=\theta e(T, V(D_i))+(1-\theta)\,e(S, V(D_i))-1.$$

We shall show that $\Delta(D_i)\geqslant 0$ for all $i$, $1\leqslant i\leqslant m$. If it is proved, then $\delta(S, T)\geqslant 0$, which is a desired contradiction, and we conclude that $H$ has a $(g,f)$-factor. If $e(T, V(D_i))\geqslant 1$ and $e(S, V(D_i))\geqslant 1$, then $\Delta(D_i)\geqslant 0$. Suppose $e(T, V(D_i))=0$. By the congruence expression (1), we have $|V(D_i)|\equiv 1 \pmod 2$. Since

$$\sum_{x\in V(D_i)} d_H(x)=2\,|E(D_i)|+e(S\cup T, V(D_i))=(2r+1)\,|V(D_i)|, \quad (3)$$

we have $e(S, V(D_i))\equiv |V(D_i)|\equiv 1 \pmod 2$. By the inequality (2), we can get $e(S, V(D_i))\geqslant 3$. Therefore $\Delta(D_i)\geqslant(1-\theta)\times 3-1\geqslant 0$. Next assume

$e(S, V(D_i))=0$. By (3), we have $e(T, V(D_i)) \equiv |V(D_i)| \pmod 2$. But this congruence expression contradicts (1). Hence this case does not occur.

Consequently, $H$ has a $\{2k+1, 1, 0\}$-factor $F$ with the property that $d_F(x)=2k+1$ if $d_H(x)=2r+1$, and $d_F(w) \in \{1, 0\}$ for all end vertices $w$.

We can now construct a $[2k, 2k+1]$-factor of $G$ each component of which is regular as follows. Let $F$ be a $\{2k+1, 1, 0\}$-factor of $H$ given above. If $d_F(w_i)=0$, then we take a $2k$-factor $R(i)$ of $C_i$ by Lemma 5. If $d_F(w_i)=1$, then we take a $[2k, 2k+1]$-factor $R(i)$ of $C_i$ such that $d_{R(i)}(w_i)=2k$ and $d_{R(i)}(x)=2k+1$ for all $x \in V(C_i)\backslash\{w_i\}$ by Lemma 5. Then the union $F \cup R(1) \cup \cdots \cup R(n)$ is a $[2k, 2k+1]$-factor of $G$ with regular components.

We next prove that if $(2r+4)/3 \leqslant 2k \leqslant 2(2r+1)/3$, then $G$ has a $[2k-1, 2k]$-factor each of whose components is regular. Define a subgraph $H$ of $G$ as above. Then $H$ has a $\{2k-1, 1, 0\}$-factor $F'$ such that $d_{F'}(x)=2k-1$ if $d_H(x)=2r+1$, and $d_{F'}(w) \in \{1, 0\}$ for all end vertices $w$. By using Lemma 5, we can construct a $[2k-1, 2k]$-factor of $G$ with regular components from $F'$ in the same way. Consequently, the proof of this lemma is complete.

*Proof of Theorem* 1. We prove the theorem by induction on $2r+1$. Let $G$ be a $(2r+1)$-regular graph and $k$ be an integer such that $2 \leqslant k \leqslant 2(2r+1)/3$. Note that every regular graph has a $[0, 1]$-factor with regular components since it has a 0-factor. By Lemma 4, we may assume that $G$ is not 2-edge-connected. Suppose $G$ has exactly one bridge $vw$. Then each component $C$ of $G-vw$ is a 2-edge-connected $[2r, 2r+1]$-graph possessing one vertex of degree $2r$. Thus $C$ has a $k$-factor or a $(k-1)$-factor by Lemma 5. Therefore $G$ has a $k$-factor or a $(k-1)$-factor, and the theorem holds. Consequently, we may assume $G$ has at least two bridges.

By Lemma 6, a 3-regular graph with at least two bridges has a $[1, 2]$-factor with regular components. Hence every 3-regular graph has a $[1, 2]$-factor with regular components, and so the theorem is true if $2r+1=3$. Similarly, we can show that every 5-regular graph has a $[2, 3]$-factor $F_1$ with regular components. Since each 3-regular component of $F_1$ has a $[1, 2]$-factor with regular components, $F_1$ has a $[1, 2]$-factor with regular components, which is of course a desired $[1, 2]$-factor of $G$. Hence the theorem follows for $2r+1=5$. In general, if a $(2r+1)$-regular graph $G$ has an $[h-1, h]$-factor $F_2$ with regular components and if each component of $F_2$ has a $[k-1, k]$-factor with regular components, then $G$ has a $[k-1, k]$-factor with regular components. By this argument, we can verify that if $2r+1 \leqslant 17$, then the theorem holds. Suppose $2r+1 \geqslant 19$. If $(2r+4)/3 \leqslant k \leqslant 2(2r+1)/3$, then a $(2r+1)$-regular graph $G$ has a $[k-1, k]$-factor with regular components by Lemma 6. Hence we may assume $k < (2r+4)/3$. Let $h$ be the greatest integer not exceeding

$2(2r+1)/3$. Then $h \geqslant 4r/3$, and $G$ has an $[h-1, h]$-factor $F$ with regular components. Since $2(h-1)/3 \geqslant 2(4r-3)/9$ and $2r+1 \geqslant 19$, we have $2(h-1)/3 \geqslant (2r+4)/3 > k$. Hence each component of $F$ has a $[k-1, k]$-factor with regular components by Lemma 1 or by the inductive hypothesis. Therefore $G$ has a $[k-1, k]$-factor with regular components, and we conclude that the proof of Theorem 1 is complete.

*Proof of Theorem* 2. Let $k$ and $r$ be positive integers such that $2r+2-\sqrt{2r+2} < 2k \leqslant 2r$. Let $k'$ be an odd integer that is one of $\{2k-1, 2k+1\}$ and not equal to $2r+1$. Let $K_{2r+3}$ denote the complete graph with vertex set $\{a_1, ..., a_{2r+3}\}$. We obtain the graph $R$ from $K_{2r+3}$ by deleting edges $a_1a_2, a_1a_3, ..., a_1a_{2r-2k+5}, a_{2r-2k+6}a_{2r-2k+7}, ..., a_{2r+2}a_{2r+3}$. It is clear that $d_R(a_1) = 2k-2$ and $d_R(a_i) = 2r+1$ for all $i$, $2 \leqslant i \leqslant 2r+3$. Let $R(1), ..., R(2r)$ be copies of $R$, and let $b_i$ be the vertex of $R_i$ whose degree is $2k-2$ for all $i$. We construct a graph $H$ with vertex set $V(R(1)) \cup \cdots \cup V(R(2r)) \cup \{c_1, ..., c_{2r-2k+2}, v\}$ as follows. Join every $b_i$ to all $c_j$ $(1 \leqslant j \leqslant 2r-2k+2)$ and $v$ by new edges, and add new edges $c_1c_2, c_3c_4, ..., c_{2r-2k+1}c_{2r-2k+2}$ (see Fig. 1). Then $d_H(v) = 2r$ and $d_H(x) = 2r+1$ for all $x \in V(H) \setminus \{v\}$.

Let $H_1, ..., H_{2r+1}$ be copies of $H$, and let $v_i$ be the vertex of $H_i$ whose degree is $2r$ for every $i$. We now construct a $(2r+1)$-regular graph $G$ as follows, which has the required property. Set $V(G) = V(H_1) \cup \cdots \cup V(H_{2r+1}) \cup \{w\}$, and join each $v_i$ to $w$ by a new edge. We shall prove that $G$ has no $\{2k, k'\}$-factor each component of which is regular.

Suppose $G$ has a $\{2k, k'\}$-factor with regular components. Then it is obvious that some $H_i$ has a $\{2k, k'\}$-factor with regular components. Thus it suffices to show that $H$ cannot have such a factor.

Assume $H$ has a $\{2k, k'\}$-factor $F$ with regular components. Since the order of $H$ is odd, $F$ has at least one $2k$-regular component $C$ of odd order. Suppose $V(C) \cap V(R(i)) \neq \varnothing$ for some $i$, $1 \leqslant i \leqslant 2r$. Since $d_H(b_i) - d_{R(i)}(b_i) = 2r-2k+3 < 2k$, $C$ contains at least one vertex in $V(R(i)) \setminus \{b_i\}$, and so $|V(C) \cap V(R(i))| \geqslant 2k+1$. If $C$ does not contain a vertex $x$ in $V(R(i)) \setminus \{b_i\}$, then $d_F(x) \leqslant 2r+3-(2k+1) < 2k+1$, a contradiction. Hence $V(R(i)) \setminus \{b_i\} \subset V(C)$. If $b_i \notin V(C)$, then $C$ is of even order, a con-

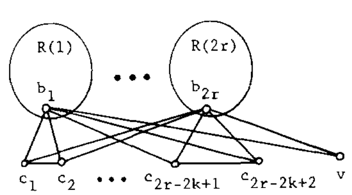


FIG. 1. The graph $H$.

tradiction. Therefore, $V(C) \cap V(R(i)) \neq \emptyset$ implies that $V(C) \supset V(R(i))$ and that $b_i$ is joined to at least one vertex, say $u$, in $\{c_1, ..., c_{2r-2k+2}, v\}$ by an edge of $C$ (since $d_{R(i)}(b_i) = 2k-2 < d_C(b_i)$). For each vertex $u_2$ in $\{c_1, ..., c_{2r-2k+2}, v\} \backslash \{u_1\}$, we have $d_F(u_1) + d_F(u_2) \geqslant 4k-1 > 2r+2$, and so $u_1$ and $u_2$ are adjacent to a common $b_t$. Hence $\{c_1, ..., c_{r-2k+2}, v\} \subset V(C)$. If $V(C) \cap V(R(j)) = \emptyset$ for some $j$, $1 \leqslant j \leqslant 2r$, then $R(j)$ contains a $2k$-regular component $C'$ of $F$ since $R(j)$ is of odd order. By the same argument as above, it follows that $V(C') \supset V(R(j))$ and $b_j$ is joined to a vertex in $\{c_1, ..., c_{2r-2k+2}, v\}$ be an edge of $C'$, which is contrary to $\{c_1, ..., c_{2r-2k+2}, v\} \subset V(C)$. Consequently, $C$ is equal to $F$, in particular, $F$ is a $2k$-factor of $H$.

On the other hand, we can show that $H$ has no $2k$-factor by Lemma 2, which establishes the proof of Theorem 2. Define two functions $g$ and $f$ on $V(H)$ by $g(x) = f(x) = 2k$ for all $x \in V(H)$, and set $S = \emptyset$ and $T = \{c_1, ..., c_{2r-2k+2}, v\}$. Since $R(1), ..., R(2r)$ are the components of $H - (S \cup T)$ that satisfy the conditions on $h(S, T)$ in Lemma 2, we obtain

$$\begin{aligned}\delta(S, T) &= (2r+1-2k)(|T|-1) + 2r - 2k - h(S, T)\\ &= (2r+1-2k)(2r-2k+3) - 2r - 1 < 0\end{aligned}$$

as $2r+2-\sqrt{2r+2} < 2k$. Therefore $H$ has no $2k$-factors.

## 3. $[a, b]$-Factors of Graphs

Katerinis [8] recently obtained the following result. Let $m$, $n$, and $k$ be positive odd integers such that $m < k < n$. If a graph $G$ has an $m$-factor and an $n$-factor, then $G$ has a $k$-factor. We give a similar result on $[a, b]$-factors. Note that the inequality (4) in Theorem 3 is equivalent to the condition that the point $(m, n)$ in the plane lies above the straight line passing through the points $(a, b)$ and $(c, d)$.

THEOREM 3. *Let $a$, $b$, $c$, $d$, $m$, $n$ be integers satisfying $0 \leqslant a < m < c$, $a \leqslant b$, $c \leqslant d$, and $m+1 \leqslant n$. Suppose a graph $G$ has both an $[a, b]$-factor and a $[c, d]$-factor. If*

$$(n-b)(m-c) \leqslant (n-d)(m-a), \tag{4}$$

*then $G$ has an $[m, n]$-factor.*

*Proof.* By Lemma 2 with $g(x) = m$ and $f(x) = n$, it suffices to show $\delta(S, T) \geqslant 0$ for all disjoint subsets $S$ and $T$ of $V(G)$, where

$$\delta(S, T) = \sum_{t \in T} (d_G(t) - m) + n\,|S| - e(S, T).$$

Note that $h(S, T) = 0$ as $m + 1 \leqslant n$. Suppose $\delta(S, T) < 0$ for some $S, T \subset V(G)$, $S \cap T = \varnothing$. Since $G$ has an $[a, b]$-factor, we have

$$\delta_1(S, T) = \sum_{t \in T} (d_G(t) - a) + b\,|S| - e(S, T) - h_1(S, T) \geqslant 0.$$

Similarly, since $G$ has a $[c, d]$-factor, we have

$$\delta_2(S, T) = \sum_{t \in T} (d_G(t) - c) + d\,|S| - e(S, T) - h_2(S, T) \geqslant 0.$$

Then we obtain

$$0 > \delta(S, T) - \delta_1(S, T) \geqslant (a - m)\,|T| + (n - b)\,|S| \tag{5}$$

and

$$0 > \delta(S, T) - \delta_2(S, T) \geqslant (c - m)\,|T| + (n - d)\,|S|. \tag{6}$$

If $S = \varnothing$, then $(c - m)\,|T| < 0$ from (6), a contradiction. Hence $S \neq \varnothing$. By (5) and (6), we get

$$\frac{n-b}{m-a} < \frac{|T|}{|S|} \qquad \text{and} \qquad \frac{|T|}{|S|} < \frac{n-d}{m-c}$$

and thus $(n-b)(m-c) > (n-d)(m-a)$, which is contrary to (4). Consequently, $\delta(S, T) \geqslant 0$ for all $S, T \subset V(G)$, $S \cap T = \varnothing$. Thus $G$ has an $[m, n]$-factor.

COROLLARY. *Let $0 < k < r$. Then every $[r-1, r]$-graph has a $[k-1, k]$-factor.*

*Proof.* Put $a = b = 0$, $c = r - 1$, $d = r$, $m = k - 1$, and $n = k$. Then (4) in Theorem 3 follows. Hence the corollary holds.

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